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ASYMPTOTIC APPROXIMATION FOR THE ELASTIC NORMAL MODES IN A STRATIFIED SOLID MEDIUM

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ABSTRACT

The asymptotic approximation method previously applied to the case of compressional wave propagation in an inhomogeneous fluid medium is carried through for the case of a solid medium. Although the method is based on the assumption of continuous variation of the elastic properties, a comparison of the dispersion curves computed by the approximate theory with those computed by the exact theory for a medium made up of two or three homogeneous layers indicates that the approximate theory is fairly accurate for the normal modes of higher order than the fifth or sixth. The approximation fails for the modes of lowest order, which are those of greatest seismological interest, but even in this case the asymptotic theory, when used in conjunction with the limiting forms of the exact theory, has some value for the purpose of rough estimation.

ASYMPTOTIC APPROXIMATION FOR THE
ELASTIC NORMAL MODES IN A STRATIFIED SOLID MEDIUM

1. INTRODUCTION

In a previous paper,[1] expressions were derived for the dispersion of the normal modes of elastic waves in a semi-infinite solid medium made up of a finite number of homogeneous parallel layers of different densities and elastic constants. The numerical computation of the phase velocity vs frequency curves from these expressions is extremely laborious if there are more than two layers and almost prohibitively so if there are more than three. In discussing the propagation of seismic surface waves in the earth's crustal layers, particularly in the continents, it may become necessary to employ more complex models than those that have been treated previously. It is therefore of interest to consider the applicability to seismic wave propagation problems of the asymptotic approximation methods that have been applied to fluid media with continuously varying elastic parameters.[2,3]

There is one serious limitation that should be pointed out at the outset. The approximations that we shall use are based on the assumption that the properties of the medium are *slowly* varying functions of one coordinate in the sense that the relative variation within a wavelength is small. For the frequencies of greatest seismological interest (periods from about 0.5 second to something over a minute), the assumption is certainly not correct for the earth's crustal layers. Nevertheless it may be of some interest to carry the analysis through for the case of a medium of slowly varying properties and then compare the resulting normal-mode dispersion curves with those obtained from the exact theory for discrete homogeneous layers.

2. SEPARATION OF THE EQUATIONS OF MOTION IN CYLINDRICAL COORDINATES

In cylindrical coordinates the equations of motion are

$$\rho \partial^2 u_r / \partial t^2 = \partial T_{rr}/\partial r + (T_{rr} - T_{\theta\theta})/r + \partial T_{r\theta}/r \partial \theta + \partial T_{rz}/\partial z \tag{1}$$

$$\rho \partial^2 u_z / \partial t^2 = \partial T_{rz}/\partial r + T_{rz}/r + \partial T_{z\theta}/r \partial \theta + \partial T_{zz}/\partial z \tag{2}$$

$$\rho \partial^2 u_\theta / \partial t^2 = \partial T_{r\theta}/\partial r + 2T_{r\theta}/r + \partial T_{\theta\theta}/r \partial \theta + \partial T_{z\theta}/\partial z , \tag{3}$$

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where the components of the stress tensor are given by

$$T_{rr} = \lambda \delta + 2\mu \, \partial u_r / \partial r \tag{4}$$

$$T_{zz} = \lambda \delta + 2\mu \, \partial u_z / \partial z \tag{5}$$

$$T_{\theta\theta} = \lambda \delta + 2\mu (\partial u_\theta / r \, \partial \theta + u_r / r) \tag{6}$$

$$T_{rz} = \mu(\partial u_z / \partial r + \partial u_r / \partial z) \tag{7}$$

$$T_{r\theta} = \mu(\partial u_\theta / \partial r - u_\theta / r + \partial u_r / r \, \partial \theta) \tag{8}$$

$$T_{z\theta} = \mu(\partial u_\theta / \partial z + \partial u_z / r \, \partial \theta) \tag{9}$$

and

$$\delta = \partial u_r / \partial r + u_r / r + \partial u_\theta / r \, \partial \theta + \partial u_z / \partial z \, . \tag{10}$$

The Lamé elastic moduli λ and μ and the density ρ are assumed to be function of the axial coordinate z, only. Following Sezawa's[5] solution of these equations for the case of a homogeneous medium, we start with particular periodic solutions in the form

$$u_r = \omega^{-2} \exp(i\omega t) \cos n\theta \, [F_1(z) \, dJ_n(kr)/dr - F_3(z) \, n \, J_n(kr)/r] \tag{11}$$

$$u_z = \omega^{-2} \exp(i\omega t) \cos n\theta \, F_2(z) \, J_n(kr) \tag{12}$$

$$u_\theta = \omega^{-2} \exp(i\omega t) \sin n\theta \, [F_3(z) \, d J_n(kr)/dr - F_1(z) \, n \, J_n(kr)/r] \tag{13}$$

where n may be any positive integer (including zero), ω is the angular frequency, k is an arbitrary number which we shall eventually use as a parameter of integration, and F_1, F_2, and F_3 are functions of z to be determined. Since we shall follow Lamb's[6] method of developing a point source solution by integration of particular solutions of the above form, the Bessel functions of the first kind $J_n(kr)$, rather than a linear combination of functions of the first and second kinds, are the appropriate radial functions in this case. An independent set of particular solutions may be obtained by substituting $\sin n\theta$ for $\cos n\theta$ in Eqs. (11) and (12) and $-\cos n\theta$ for $\sin n\theta$ in Eq. (13). However, since we shall not be concerned with the problem of representing particular types of point sources by summation over the azimuthal characteristic number, n, the set given by Eqs. (11), (12), and (13) is sufficiently general for our purpose.

By substituting these expressions for the displacements in the equations of motion, we obtain the following equations for the functions F_1, F_2, and F_3

$$\mu \ddot{F}_1 + \dot{\mu} \dot{F}_1 + [\rho \omega^2 - (\lambda + 2\mu) k^2] F_1 + (\lambda + \mu) \dot{F}_2 + \dot{\mu} F_2 = 0 \qquad (14)$$

$$(\lambda + 2\mu) \ddot{F}_2 + (\dot{\lambda} + 2\dot{\mu}) \dot{F}_2 + [\rho \omega^2 - \mu k^2] F_2 - (\lambda + \mu) k^2 \dot{F}_1 - \dot{\lambda} k^2 F_1 = 0 \qquad (15)$$

$$\mu \ddot{F}_3 + \dot{\mu} \dot{F}_3 + [\rho \omega^2 - \mu k^2] F_3 = 0 \qquad (16)$$

where the dot denotes differentiation with respect to z. If λ, μ, and ρ were constants, the substitutions

$$F_1 = \dot{Z}_2 - Z_1 \qquad (17)$$

$$F_2 = k^2 Z_2 - \dot{Z}_1 \qquad (18)$$

satisfy Eqs. (14) and (15) if the functions Z_1, and Z_2 satisfy the equations

$$\ddot{Z}_1 + [\rho \omega^2/(\lambda + 2\mu) - k^2] Z_1 = 0 \qquad (19)$$

$$\ddot{Z}_2 + [\rho \omega^2/\mu - k^2] Z_2 = 0 . \qquad (20)$$

In this case the terms in Z_1, represent waves traveling with the compressional wave velocity, $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$, and the terms in Z_2 and F_3 (which satisfies the same differential equation as Z_2) represent waves traveling with the rotational wave velocity, $\beta = [\mu/\rho]^{1/2}$.

Equations (17) through (20) are still approximately valid when λ, μ, and ρ are functions of z, provided $\dot{\mu}/\mu k$, $\dot{\lambda}/\lambda k$, and $\dot{\rho}/\rho k$ are sufficiently small to be ignored. This means essentially that the relative variations of the properties of the medium within a wavelength must be very small. When this is not the case, it will not, in general, be possible to make a clear distinction between the two types of waves. In other words, in an inhomogeneous medium there will be a coupling between compressional and rotational waves at every point of the medium for which $\dot{\mu}/\mu k$, etc., are not negligible.

Making the substitution of Eqs. (17) and (18) and dropping the common time factor, $\exp(i\omega t)$, the elementary displacements become

$$u_r = \omega^{-2} \cos n\theta [(\dot{Z}_2 - Z_1) d J_n (kr)/dr - F_3 n J_n (kr)/r] \qquad (21)$$

$$u_z = \omega^{-2} \cos n\theta [k^2 Z_2 - \dot{Z}_1] J_n (kr) \qquad (22)$$

$$u_\theta = \omega^{-2} \sin n\theta [F_3 d J_n (kr)/dr - (\dot{Z}_2 - Z_1) n J_n (kr)/r] , \qquad (23)$$

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and using Eqs. (19) and (20), the stress components across a horizontal plane become

$$T_{zz} = \omega^{-2} \cos n\theta \left[(\rho\omega^2 - 2\mu k^2) Z_1 + 2\mu k^2 \dot{Z}_2 \right] \tag{24}$$

$$T_{rz} = \omega^{-2} \cos n\theta \left[\{ (2\mu k^2 - \rho\omega^2) Z_2 - 2\mu \dot{Z}_1 \} \, dJ_n(kr)/dr - \mu \dot{F}_3 \, n J_n(kr)/r \right] \tag{25}$$

$$T_{\theta z} = \omega^{-2} \sin n\theta \left[\mu \dot{F}_3 \, dJ_n(kr)/dr - \{ (2\mu k^2 - \rho\omega^2) Z_2 - 2\mu \dot{Z}_1 \} n J_n(kr)/r \right] . \tag{26}$$

At large values of kr, $J_n(kr)/r$ becomes small compared to $J_n(kr)$ and $dJ_n(kr)/dr$, so that the terms in F_3 contribute appreciably only to u_θ and the terms in Z_1 and Z_2 only to u_r and u_z. Thus at large distances the F_3 terms represent horizontally polarized transverse (SH) waves, the Z_2 terms represent transverse waves polarized in a vertical plane (SV), and the Z_1 terms represent longitudinal (P) waves.

3. BOUNDARY CONDITIONS

Solutions representing a point source at $r = 0$, $z = h$, may be obtained by requiring the stress components to be continuous across the plane $z = h$, but allowing u_r, u_z and u_θ to have discontinuities such that, when the elementary solutions are integrated with respect to k over the interval zero to infinity, the discontinuities disappear except for a singularity at the source. The other boundary conditions are that the stress components T_{zz}, T_{rz}, $T_{\theta z}$ shall vanish at the free surface ($z = 0$) and at large values of z the functions Z_1, Z_2 and F_3 must be of such a form as to represent, with the time factor $\exp(i\omega t)$, only waves propagated or exponentially damped in the $+z$ direction.

Introducing the abbreviation $\gamma(z) = 2\mu k^2/\rho\omega^2 = 2(\beta k/\omega)^2$, the stress boundary conditions at the free surface and at the source are

$$[\gamma(0) - 1] Z_1(0) - \gamma(0) \dot{Z}_2(0) = 0 \tag{27}$$

$$[\gamma(0) - 1] Z_2(0) - \gamma(0) \dot{Z}_1(0)/k^2 = 0 \tag{28}$$

$$\dot{F}_3(0) = 0 \tag{29}$$

$$[\gamma(h) - 1] \Delta Z_1 - \gamma(h) \Delta \dot{Z}_2 = 0 \tag{30}$$

$$[\gamma(h) - 1] \Delta Z_2 - \gamma(h) \Delta \dot{Z}_1/k^2 = 0 \tag{31}$$

$$\Delta \dot{F}_3 = 0 \tag{32}$$

where Δ signifies the discontinuity in any quantity at $z = h$.

Let $M_\alpha(k, z)$ and $N_\alpha(k,z)$ be two linearly independent solutio⸱ of Eq. (19) which, with exp $(i\omega t)$, represent waves propagating in the negative and positive z directions respectively for large values of z. Let $M_\beta(k,z)$ and $N_\beta(k,z)$ be the two corresponding solutions of Eq. (20). The boundary conditions at the free surface, the source, and at infinity may then be satisfied by solutions of the form

$$0 < z < h \quad Z_1 = A M_\alpha + B N_\alpha \tag{33}$$

$$Z_2 = C M_\beta + D N_\beta \tag{34}$$

$$F_3 = E M_\beta + F N_\beta \tag{35}$$

$$h < z \quad Z_1 = G N_\alpha \tag{36}$$

$$Z_2 = H N_\beta \tag{37}$$

$$F_3 = I N_\beta . \tag{38}$$

By using Eqs. (30) and (31) and integrating with respect to k from zero to infinity, the discontinuities in the displacements at $z = h$ become

$$\Delta u_r = -\omega^{-2} \cos n\theta \int_0^\infty [\{\Delta Z_1 / \gamma(h)\} \, d J_n(kr)/dr + \Delta F_3 \, n J_n(kr)/r] \, dk \tag{39}$$

$$\Delta u_z = \omega^{-2} \cos n\theta \int_0^\infty \{\Delta Z_2 / \gamma(h)\} J_n(kr) \, k^2 \, dk \tag{40}$$

$$\Delta u_\theta = \omega^{-2} \sin n\theta \int_0^\infty [\Delta F_3 \, d J_n(kr)/dr + \{\Delta Z_1 / \gamma(h)\} n J_n(kr)/r] \, dk . \tag{41}$$

If we set

$$\Delta Z_1 = -\omega^2 \gamma(h) \, k^{n+1} P_n \tag{42}$$

$$\Delta Z_2 = \omega^2 \gamma(h) \, k^{n-1} S_n \tag{43}$$

$$\Delta F_3 = \omega^2 \, k^{n+1} Q_n \tag{44}$$

where P_n, S_n, Q_n are constants, the discontinuities become

$$\Delta u_r = \cos\, n\,\theta\, [P_n\,(d/dr) - n\,Q_n/r]\int_0^\infty J_n\,(kr)\,k^{n+1}\,dk$$

$$\Delta u_z = \cos\, n\,\theta\, S_n \int_0^\infty J_n\,(kr)\,k^{n+1}\,dk$$

$$\Delta u_\theta = \sin\, n\,\theta\, [Q_n\,(d/dr) - n\,P_n/r]\int_0^\infty J_n\,(kr)\,k^{n+1}\,dk\ .$$

The integral in these expressions, considered as the limit of $\int_0^\infty e^{-ak}\,J_n\,(kr)\,k^{n+1}\,dk$ as a approaches zero, vanishes everywhere except at $r = 0$, where it has an essential singularity, thus satisfying the point source requirements.

From Eqs. (33) to (38) we have

$$\Delta Z_1 = (G - B)\,N_\alpha(h) - A\,M_\alpha(h) \tag{45}$$

$$\Delta Z_2 = (H - D)\,N_\beta(h) - C\,M_\beta(h) \tag{46}$$

$$\Delta F_3 = (I - F)\,N_\beta(h) - E\,M_\beta(h)\ . \tag{47}$$

By using Eqs. (27) through (32) with (42) through (47), the coefficients in Z_1, Z_2 and F_3 may be determined as follows

$$A = \omega^2\,k^{n+1}\,b_\alpha^{-1}\,[S_n\,\{\gamma(h) - 1\}\,N_\alpha(h) + P_n\,\gamma(h)\,\dot{N}_\alpha(h)] \tag{48}$$

$$C = -\,\omega^2\,k^{n+1}\,b_\beta^{-1}\,[P_n\,\{\gamma(h) - 1\}\,N_\beta(h) + k^{-2}\,S_n\,\gamma(h)\,\dot{N}_\beta(h)] \tag{49}$$

$$B = \omega^2\,k^{n+1}\,f^{-1}(k)\,[P_n\,W(k) + S_n\,X(k)] \tag{50}$$

$$D = -\,\omega^2\,k^{n+1}\,f^{-1}(k)\,[P_n\,Y(k) + S_n\,Z(k)] \tag{51}$$

$$E = -\,\omega^2\,k^{n+1}\,b_\beta^{-1}\,Q_n\,\dot{N}_\beta(h) \tag{52}$$

$$F = \omega^2\,k^{n+1}\,b_\beta^{-1}\,Q_n\,\dot{M}_\beta(0)\,\dot{N}_\beta(h)/\dot{N}_\beta(0) \tag{53}$$

$$G = B + \omega^2\,k^{n+1}\,b_\alpha^{-1}\,[P_n\,\gamma(h)\,\dot{M}_\alpha(h) + S_n\,\{\gamma(h) - 1\}\,M_\alpha(h)] \tag{54}$$

$$H = D - \omega^2\,k^{n+1}\,b_\beta^{-1}\,[P_n\,\{\gamma(h) - 1\}\,M_\beta(h) + k^{-2}\,S_n\,\gamma(h)\,\dot{M}_\beta(h)] \tag{55}$$

$$I = F - \omega^2 k^{n+1} Q_a \dot{M}_\beta(h) b_\beta^{-1} . \tag{56}$$

The new symbols introduced in these expressions are

$$b_\alpha = M_\alpha \dot{N}_\alpha - \dot{M}_\alpha N_\alpha \tag{57}$$

$$b_\beta = M_\beta \dot{N}_\beta - \dot{M}_\beta N_\beta \tag{58}$$

$$f(k) = [\gamma(0) - 1]^2 N_\alpha(0) N_\beta(0) - [\gamma(0)/k]^2 \dot{N}_\alpha(0) \dot{N}_\beta(0) \tag{59}$$

$$W(k) = b_\alpha^{-1} [\gamma(0) \{\gamma(0) - 1\}\{\gamma(h) - 1\} N_\beta(h) b_\alpha + \{\gamma(0)/k\}^2 \gamma(h) \dot{M}_\alpha(0) \dot{N}_\alpha(h) \dot{N}_\beta(0)$$
$$- \{\gamma(0) - 1\}^2 \gamma(h) M_\alpha(0) \dot{N}_\alpha(h) N_\beta(0)] \tag{60}$$

$$X(k) = b_\alpha^{-1} [\gamma(0) \{\gamma(0) - 1\} \gamma(h) k^{-2} \dot{N}_\beta(h) b_\alpha + \{\gamma(0)/k\}^2 \{\gamma(h) - 1\} \dot{M}_\alpha(0) N_\alpha(h) \dot{N}_\alpha(0)$$
$$- \{\gamma(0) - 1\}^2 \{\gamma(h) - 1\} M_\alpha(0) N_\alpha(h) N_\beta(0)] \tag{61}$$

$$Y(k) = b_\beta^{-1} [\gamma(0)\{\gamma(0) - 1\}\gamma(h) k^{-2} \dot{N}_\alpha(h) b_\beta + \{\gamma(0)/k\}^2 \{\gamma(h) - 1\} \dot{M}_\beta(0) N_\beta(h) \dot{N}_\alpha(0)$$
$$- \{\gamma(0) - 1\}^2 \{\gamma(h) - 1\} M_\beta(0) N_\beta(h) N_\alpha(0)] \tag{62}$$

$$Z(k) = b_\beta^{-1} k^{-2} [\gamma(0)\{\gamma(0) - 1\}\{\gamma(h) - 1\} N_\alpha(h) b_\beta + \{\gamma(0)/k\}^2 \gamma(h) \dot{M}_\beta(0) \dot{N}_\beta(h) \dot{N}_\alpha(0)$$
$$- \{\gamma(0) - 1\}^2 \gamma(h) M_\beta(0) \dot{N}_\beta(h) N_\alpha(0)] . \tag{63}$$

The quantities b_α and b_β will, in general, be functions of k, but may be shown from the differential Eqs. (19) and (20) to be independent of z. Since b_α and b_β are the Wronskians of pairs of linearly independent functions, they cannot vanish. Thus the coefficients A, C and E are finite for all finite values of k, while B, D, G and H have poles at the zeros of $f(k)$ and F and I have poles at the zeros of $\dot{N}_\beta(0)$.

4. POINT SOURCE SOLUTION AS A SUM OF NORMAL MODE SOLUTIONS

We now integrate Eqs. (21), (22) and (23) with respect to k from zero to infinity and evaluate the integrals in the form of sums of residues at the poles of the integrands by applying Lamb's[6] transformation of

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the path of integration in the complex k-plane. To simplify the resulting expressions the terms in $n J_n (kr)/r$, which become small at large values of r, will be dropped. The displacements then become

$$u_r = \pi i \cos n \theta \sum_{k_m} k_m^{n+1} [d H_n^{(2)} (k_m r)/dr] [df/dk]_{k_m}^{-1} [\dot{N}_\beta (z) \{P_n Y (k_m) + S_n Z (k_m)\}$$
$$+ N_\alpha (z) \{P_n W (k_m) + S_n X (k_m)\}] \qquad (64)$$

$$u_z = \pi i \cos n \theta \sum_{k_m} k_m^{n+1} H_n^{(2)} (k_m r) [df/dk]_{k_m}^{-1} [k_m^2 N_\beta (z)\{P_n Y (k_m) + S_n Z (k_m)\}$$
$$+ \dot{N}_\alpha (z) \{P_n W (k_m) + S_n X (k_m)\}] \qquad (65)$$

$$u_\theta = \pi i \sin n \theta \sum_{k'_m} k_m'^{n+1} [d H_n^{(2)} (k'_m r)/dr] [d \dot{N}_\beta (0)/dk]_{k'_m}^{-1} N_\beta (z) Q_n \dot{N}_\beta (h)/N_\beta (0) , \qquad (66)$$

where the k_m 's are the values of k at which $f(k) = 0$ and the k'_m 's are the values of k at which $\dot{N}_\beta (0) = 0$. At $k = k_m$ the functions W, X, Y and Z reduce to

$$W (k_m) = [\gamma (0) - 1] [\gamma (0) \{\gamma (h) - 1\} N_\beta (h) - \gamma (h)\{\gamma (0) - 1\} \dot{N}_\alpha (h) N_\beta (0)/\dot{N}_\alpha (0)] \qquad (67)$$

$$X (k_m) = [\gamma (0) - 1] [\gamma (0) \gamma (h) k^{-2} \dot{N}_\beta (h) - \{\gamma (0) - 1\}\{\gamma (h) - 1\} N_\alpha (h) N_\beta (0)/\dot{N}_\alpha (0)] \qquad (68)$$

$$Y (k_m) = [\gamma (0) - 1] [\gamma (0) \gamma (h) k^{-2} \dot{N}_\alpha (h) - \{\gamma (0) - 1\}\{\gamma (h) - 1\} N_\beta (h) N_\alpha (0)/\dot{N}_\beta (0)] \qquad (69)$$

$$Z (k_m) = [\gamma (0) - 1] k^{-2} [\gamma (0)\{\gamma (h) - 1\} N_\alpha (h) - \gamma (h) \{\gamma (0) - 1\} \dot{N}_\beta (h) N_\alpha (0)/\dot{N}_\beta (0)] . \qquad (70)$$

An expression for the velocity potential in a fluid medium in a form analogous to Eqs. (64), (65) and (66) for the case of a radially symmetrical point source ($n = 0$), has been derived in a previous paper.[3] It was shown that at sufficiently high frequencies the interference pattern produced by the superposition of a large number of normal modes leads to a distribution of amplitudes which is the same as that which would be computed on the basis of ray geometry. In particular, if the variation of velocity with z is such as to produce a geometrical shadow zone, the normal modes interfere destructively in the shadow zone. At finite frequencies, the destructive interference is not quite complete and there is a residual disturbance that is given by expressions formally similar to those occurring in the optical theories of diffraction at barriers and caustics. It does not appear that any new conclusions of much significance will emerge from carrying through the same analysis in the case under consideration. What we do wish to examine further is whether or not the asymptotic solutions of Eqs. (19) and (20) can be used to draw any conclusions concerning the dispersion characteristics of the normal modes of low order, that make the principal contribution to the low frequency surface wave phases on seismograms.
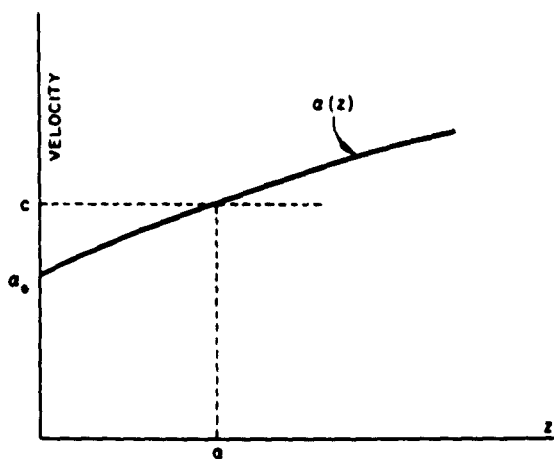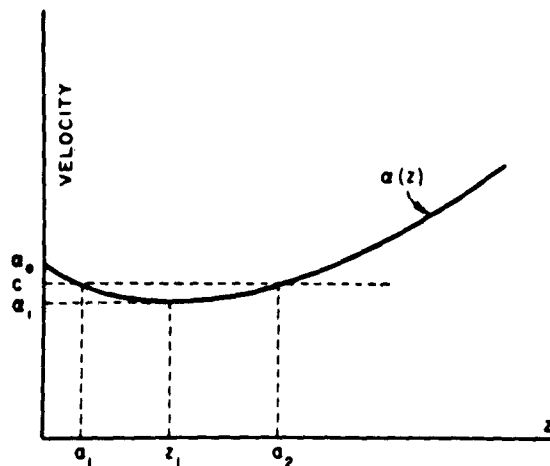
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Fig. 1. Velocity function for Case I.

Fig. 2. Velocity function for Case II.

Case I. Velocity increases monotonically with z as shown in Fig. 1

$$c < \alpha_0 \qquad N_\alpha \rightarrow (2/\pi k s_\alpha)^{\frac{1}{2}} \; \exp \left[-k y_\alpha(z) \; + \; 5\pi i/12 \right] \tag{78}$$

$$\left. \begin{array}{c} c > \alpha_0 \\ z > a \end{array} \right\} \; N_\alpha \rightarrow (2/\pi k s_\alpha)^{\frac{1}{2}} \; \exp \left[-k y_\alpha(z) \; + \; 5\pi i/12 \right] \tag{79}$$

$$\left. \begin{array}{c} c > \alpha_0 \\ z < a \end{array} \right\} \; N_\alpha \rightarrow 2 \, (2/\pi k s_\alpha)^{\frac{1}{2}} \; \exp \, (2\pi i/3) \; \cos \; \left[k \, s_\alpha(z) - \pi/4 \right] \tag{80}$$

where

$$y_\alpha(z) = \int_a^z \sqrt{1 - [c/\alpha(z)]^2} \; dz \qquad z > a \tag{81}$$

$$s_\alpha(z) = \int_z^a \sqrt{[c/\alpha(z)]^2 - 1} \; dz \qquad z < a \tag{82}$$

and a is the value of z at which $\alpha = c$. For $c < \alpha_0$, a may be set equal to zero in Eqs. (81) and (82).

Case II. Velocity has a single minimum, α_1, at $z = z_1$, as shown in Fig. 2

$$c < \alpha_1 \qquad N_\alpha \rightarrow (2/\pi k s_\alpha)^{\frac{1}{2}} \exp \left[-k y_{\alpha,2}(z) + 5\pi i/12 \right] \qquad (83)$$

$$\left.\begin{array}{c} \alpha_1 < c < \alpha_0 \\ z > a_2 \end{array}\right\} \qquad N_\alpha \rightarrow (2/\pi k s_\alpha)^{\frac{1}{2}} \exp \left[-k y_{\alpha,2}(z) + 5\pi i/12 \right] \qquad (84)$$

$$\left.\begin{array}{c} \alpha_1 < c < \alpha_0 \\ a_1 < z < a_2 \end{array}\right\} \qquad N_\alpha \rightarrow 2 (2/\pi k s_\alpha)^{\frac{1}{2}} \exp (2\pi i/3) \cos \left[k x_{\alpha,2}(z) - \pi/4 \right] \qquad (85)$$

$$\left.\begin{array}{c} \alpha_1 < c < \alpha_0 \\ z < a_1 \end{array}\right\} \qquad N_\alpha \rightarrow 2 (2/\pi k s_\alpha)^{\frac{1}{2}} \exp \left[k y_{\alpha,1}(z) + 5\pi i/12 \right] \cos \left[k x_{\alpha,2}(a_1) \right] \qquad (86)$$

$$\left.\begin{array}{c} c > \alpha_0 \\ z > a_2 \end{array}\right\} \qquad N_\alpha \rightarrow (2/\pi k s_\alpha)^{\frac{1}{2}} \exp \left[-k y_{\alpha,2}(z) + 5\pi i/12 \right] \qquad (87)$$

$$\left.\begin{array}{c} c > \alpha_0 \\ z < a_2 \end{array}\right\} \qquad N_\alpha \rightarrow 2 (2/\pi k s_\alpha)^{\frac{1}{2}} \exp (2\pi i/3) \cos \left[k x_{\alpha,2}(z) - \pi/4 \right] \qquad (88)$$

where

$$y_{\alpha,2}(z) = \int_{a_2}^{z} \sqrt{1 - [c/\alpha(z)]^2} \, dz \qquad (89)$$

$$y_{\alpha,1}(z) = \int_{z}^{a_1} \sqrt{1 - [c/\alpha(z)]^2} \, dz \qquad (90)$$

$$x_{\alpha,2}(z) = \int_{z}^{a_2} \sqrt{[c/\alpha(z)]^2 - 1} \, dz \qquad (91)$$

and for $c < \alpha_1$ we take $a_1 = a_2 = z_1$.

To the same order of approximation the derivative $\dot{N}_\alpha$ is given by

Case I

$$c < \alpha_0 \qquad \dot{N}_\alpha \rightarrow (2 k s_\alpha/\pi)^{\frac{1}{2}} \exp \left[-k y_\alpha(z) - \pi i/12 \right] \qquad (92)$$

$$\left.\begin{array}{c} c > \alpha_0 \\ z > a \end{array}\right\} \qquad \dot{N}_\alpha \rightarrow (2 k s_\alpha/\pi)^{\frac{1}{2}} \exp \left[-k y_\alpha(z) - \pi i/12 \right] \qquad (93)$$

$$\left.\begin{array}{c} c > \alpha_0 \\ z < a \end{array}\right\} \qquad \dot{N}_\alpha \rightarrow 2 (2 k s_\alpha/\pi)^{\frac{1}{2}} \exp (2\pi i/3) \sin \left[k x_\alpha(z) - \pi/4 \right] . \qquad (94)$$

Case II

$$c < \alpha_1 \qquad \dot{N}_\alpha \to (2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ [-ky_{\alpha,2}(z) - \pi i/12] \qquad (95)$$

$$\left.\begin{array}{l} \alpha_1 < c < \alpha_0 \\ z > a_2 \end{array}\right\} \qquad \dot{N}_\alpha \to (2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ [-k\,y_{\alpha,2}(z) - \pi i/12] \qquad (96)$$

$$\begin{array}{l} \alpha_1 < c < \alpha_0 \\ a_1 < z < a_2 \end{array} \qquad \dot{N}_\alpha \to 2\,(2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ (2\,\pi i/3)\sin\ [kx_{\alpha,2}(z) - \pi/4] \qquad (97)$$

$$\left.\begin{array}{l} \alpha_1 < c < \alpha_0 \\ z < a_1 \end{array}\right\} \qquad \dot{N}_\alpha \to 2\,(2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ [ky_{\alpha,1}(z) - \pi i/12]\cos\ [kx_{\alpha,2}(a_1)] \qquad (98)$$

$$\left.\begin{array}{l} c > \alpha_0 \\ z > a_2 \end{array}\right\} \qquad \dot{N}_\alpha \to (2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ [-ky_{\alpha,2}(z) - \pi i/12] \qquad (99)$$

$$\left.\begin{array}{l} c > \alpha_0 \\ z < a_2 \end{array}\right\} \qquad \dot{N}_\alpha \to 2\,(2\,ks_\alpha/\pi)^{\frac{1}{2}} \exp\ (2\,\pi i/3)\sin\ [kx_{\alpha,2}(z) - \pi/4]\ . \qquad (100)$$

The functions N_β and $\dot{N}_\beta$ are given by the same expressions with β substituted for α. Corresponding values of z at which $\beta = c$ are denoted by b. The characteristic Eq. (72) for the normal modes of Rayleigh type then takes the following forms.

Case I

$$c < \beta_0 \qquad [\gamma(0) - 1]^2 - [\gamma(0)]^2\sqrt{[1 - (c/\alpha_0)^2]\,[1 - (c/\beta_0)^2]}\ = 0 \qquad (101)$$

$$\beta_0 < c < \alpha_0 \quad [\gamma(0) - 1]^2 + [\gamma(0)]^2\sqrt{[1 - (c/\alpha_0)^2]\,[(c/\beta_0)^2 - 1]}\ \tan\ [k_m x_\beta(0) - \pi/4]\ = 0 \qquad (102)$$

$$\alpha_0 < c \qquad [\gamma(0) - 1]^2 - [\gamma(0)]^2\sqrt{[(c/\alpha_0)^2 - 1]\,[(c/\beta_0)^2 - 1]}$$
$$\tan\ [k_m x_\alpha(0) - \pi/4]\ \tan\ [k_m x_\beta(0) - \pi/4]\ = 0\ . \qquad (103)$$

Case II

$$c < \beta_1 \qquad [\gamma(0) - 1]^2 - [\gamma(0)]^2\sqrt{[1 - (c/\alpha_0)^2]\,[1 - (c/\beta_0)^2]}\ = 0 \qquad (104)$$

$$\beta_1 < c < \beta_0 < \alpha_1 \quad \left\{[\gamma(0) - 1]^2 - [\gamma(0)]^2\sqrt{[1 - (c/\alpha_0)^2]\,[1 - (c/\beta_0)^2]}\right\}\cos\ [k_m x_{\beta,2}(b_1)]\ = 0 \qquad (105)$$

$$\beta_0 < c < \alpha_1 \qquad [\gamma(0) - 1]^2 + [\gamma(0)]^2\sqrt{[1 - (c/\alpha_0)^2]\,[(c/\beta_0)^2 - 1]}$$
$$\tan\ [k_m x_{\beta,2}(0) - \pi/4]\ = 0 \qquad (106)$$

$$\alpha_1 < c < \alpha_0 \quad \left\{ [\gamma(0)-1]^2 + [\gamma(0)]^2 \sqrt{[1-(c/\alpha_0)^2][(c/\beta_0)^2-1]} \right.$$

$$\left. \tan [k_m x_{\beta,2}(0) - \pi/4] \right\} \cos [k_m x_{\alpha,2}(a_1)] = 0 \quad (107)$$

$$c > \alpha_0 \quad [\gamma(0)-1]^2 - [\gamma(0)]^2 \sqrt{[(c/\alpha_0)^2-1][(c/\beta_0)^2-1]} \tan [k_m x_{\alpha,2}(0) - \pi/4]$$

$$\tan [k_m x_{\beta,2}(0) - \pi/4] = 0 . \quad (108)$$

Equations (101), (104), and the first factor of Eq.(105) are the same as the equation for the velocity of Rayleigh waves on a homogeneous medium having velocities $\alpha = \alpha_0$ and $\beta = \beta_0$. This equation determines a single real value of the phase velocity, c, for all values of k. Actually, of course, this value of c is merely the asymptote approached by the Rayleigh wave phase velocity at high frequencies, so that, the present approximation is inadequate to deal with the dispersion of the mode of lowest order.

The characteristic Eq. (73) for the normal modes of SH type has the following asymptotic forms.

Case I

$$c < \beta_0 \qquad \text{No roots exist}$$

$$c > \beta_0 \qquad \sin [k'_m x_\beta(0) - \pi/4] = 0$$

or

$$k'_m x_\beta(0) = \pi(m + 1/4) \quad m = 0,1,2,3,.... \quad (109)$$

Case II

$$c < \beta_1 \qquad \text{No roots exist}$$

$$\beta_1 < c < \beta_0 \quad \cos [k'_m x_{\beta,2}(b_1)] = 0$$

or

$$k'_m x_{\beta,2}(b_1) = \pi(m + 1/2) \quad m = 0,1,2,3,.... \quad (110)$$

$$\beta_0 < c \qquad \sin [k'_m x_{\beta,2}(0) - \pi/4] = 0$$

or

$$k'_m x_{\beta,2}(0) = \pi(m + 1/4) \quad m = 0,1,2,3,.... \quad (111)$$

6. APPLICATION OF APPROXIMATE THEORY TO A LAYERED MEDIUM

The development of the approximate expressions for the functions N_α and N_β involves the assumption that the velocities and their first derivatives are continuous functions of z so that, in general, the present theory is not applicable to a medium made up of discrete layers. If the discontinuities in the velocities and
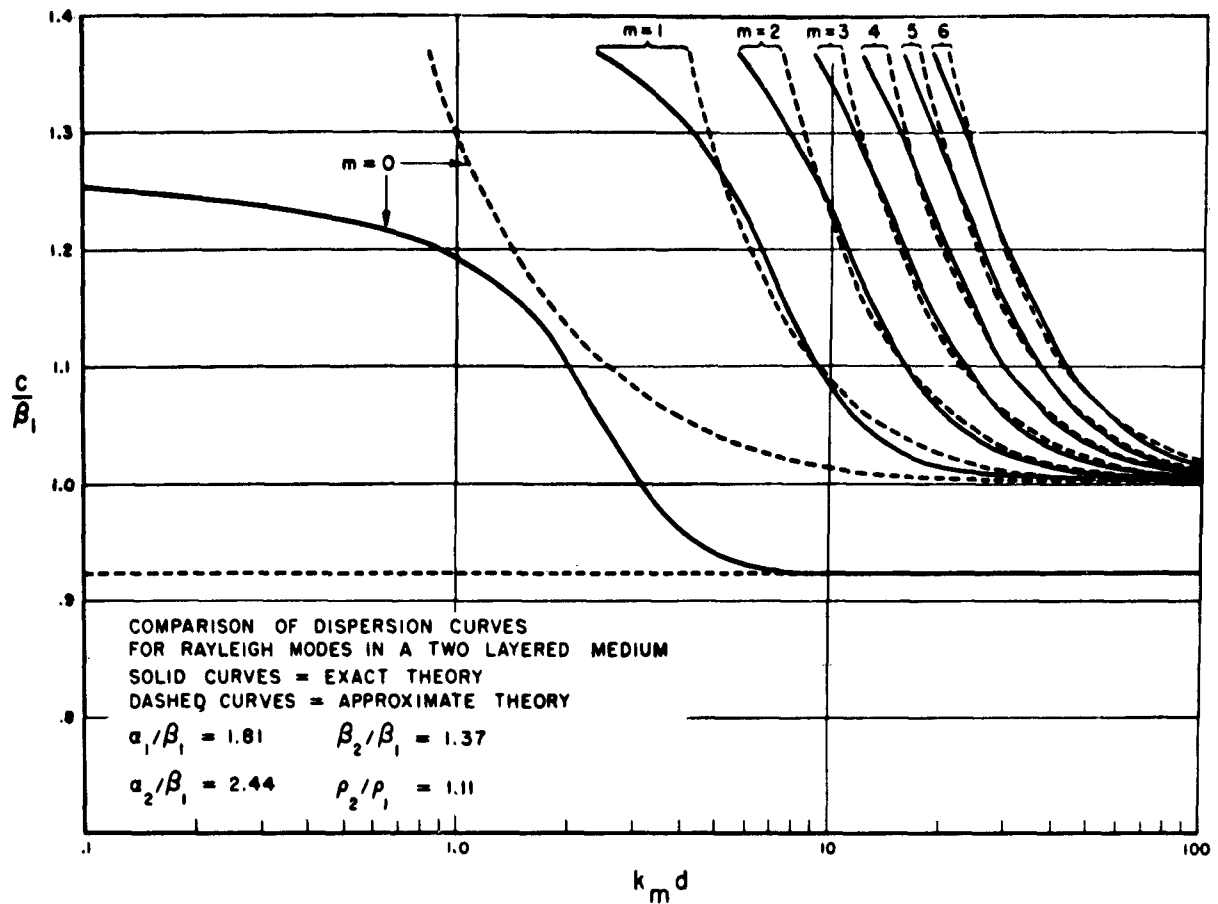
Fig. 3. Comparison of dispersion curves for Rayleigh Modes in a two layered medium.

the density are small, however, the amplitudes of the waves reflected from the boundaries will be small except for angles of incidence greater than the critical angle. If we can ignore the boundary reflections at angles less than the critical angle, there is no fundamental distinction between the cases of continuous and discontinuous velocity variation.

In the case of monotonically increasing velocity, the quantities $x_\alpha(0)$ and $x_\beta(0)$ become for discrete layers, each of constant velocity,

$$x_\alpha(0) = \sum_{i=1}^{\alpha_i < c} d_i \sqrt{(c/\alpha_i)^2 - 1} \tag{112}$$

$$x_\beta(0) = \sum_{i=1}^{\beta_i < c} d_i \sqrt{(c/\beta_i)^2 - 1} \tag{113}$$
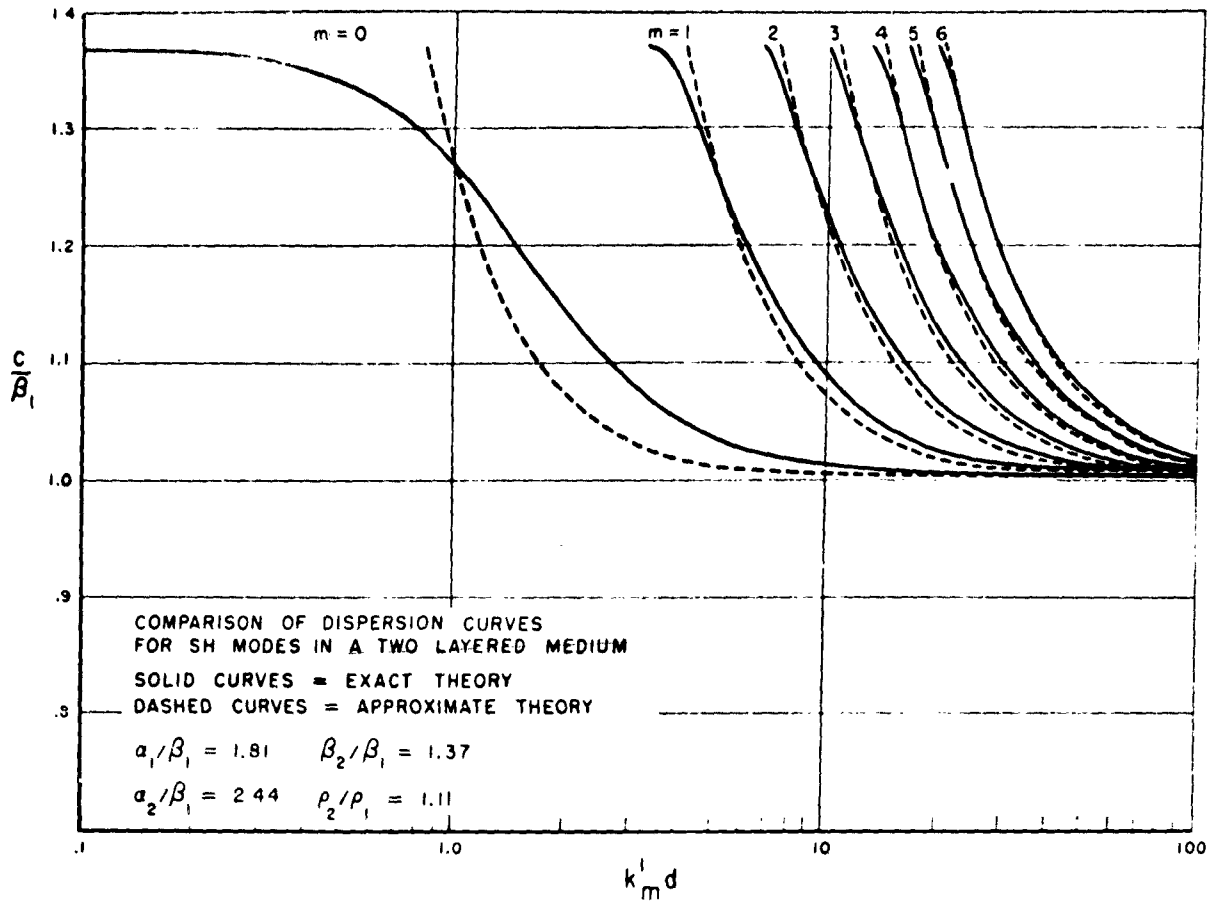
Fig. 4. Comparison of dispersion curves for SH modes in a two layered medium.

where d_i is the thickness of the ith layer and the summations are extended over all layers for which α_i or β_i respectively are less than c. If the velocity of successive layers decreases and then increases, we find the same Eqs. (112) and (113) for $x_{\alpha,2}(0)$ and $x_{\beta,2}(0)$ when $c > \alpha_0$ and $c > \beta_0$ respectively. When $c < \alpha_0$ or $c < \beta_0$, the summations extend only over the layers for which $\alpha_i < c$ or $\beta_i < c$.

Phase velocity dispersion curves have been computed according to the present approximate theory and according to the exact theory for a two-layered medium (the second being semi-infinite) having the following properties:

$$\alpha_1/\beta_1 = 1.81 \qquad \beta_2/\beta_1 = 1.37$$

$$\alpha_2/\beta_1 = 2.44 \qquad \rho_2/\rho_1 = 1.11 .$$

The results are plotted in dimensionless form, c/β_1 versus kd, for the first seven normal modes of Rayleigh type in Fig. 3, and for the normal modes of SH type in Fig. 4. The exact and approximate curves
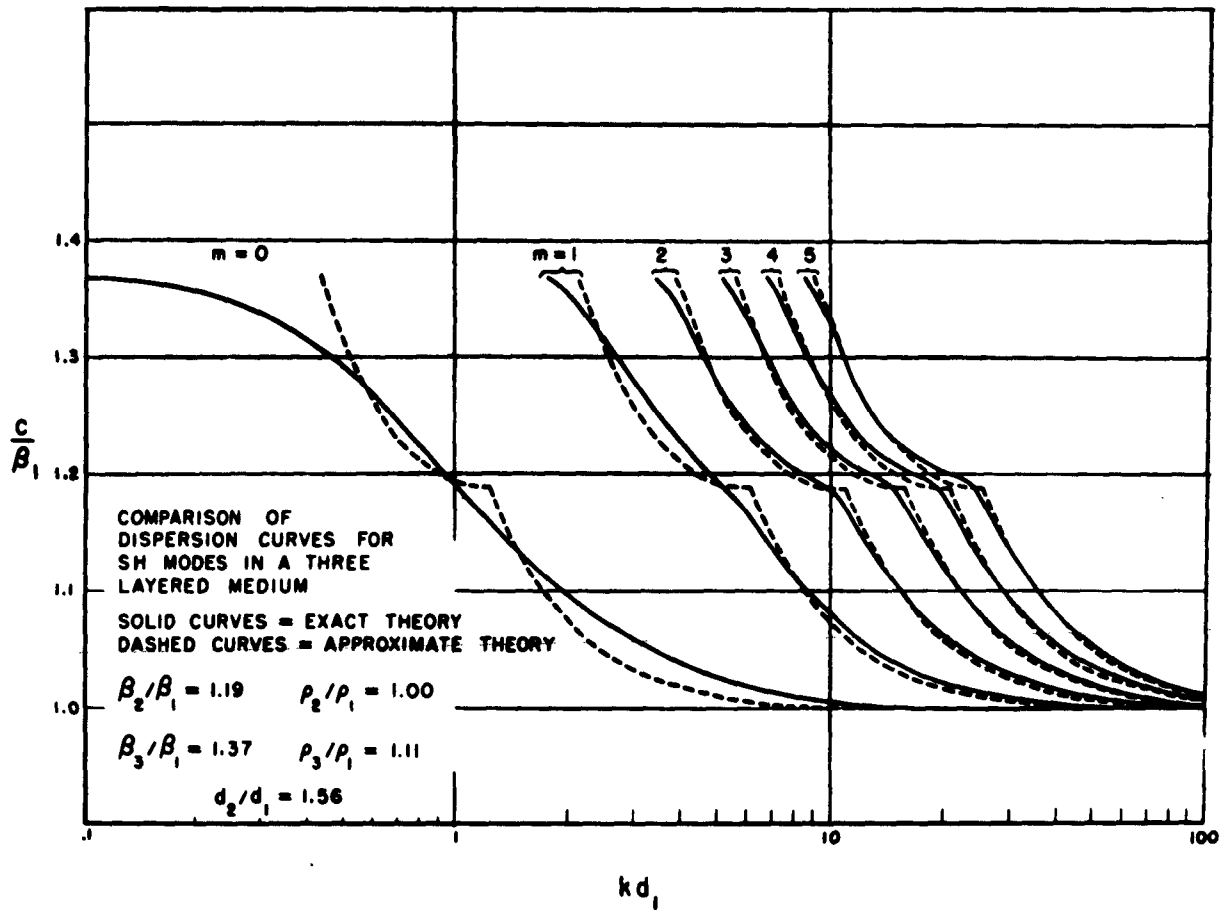
Fig. 5. Comparison of dispersion curves for *SH* modes in a three layered medium, Case I.

are compared for the *SH* modes in two 3-layered media in Figs. 5 and 6. Those shown in Fig. 5 are for the case

$$\beta_2/\beta_1 = 1.19 \qquad \rho_2/\rho_1 = 1.00 \qquad d_2/d_1 = 1.56 ,$$

$$\beta_3/\beta_1 = 1.37 \qquad \rho_3/\rho_1 = 1.11$$

and those shown in Fig. 6 illustrate the effect of a low velocity second layer

$$\beta_2/\beta_1 = .988 \qquad \rho_2/\rho_1 = 1.00 \qquad d_2/d_1 = .871 .$$

$$\beta_3/\beta_1 = 1.37 \qquad \rho_3/\rho_1 = 1.11 .$$
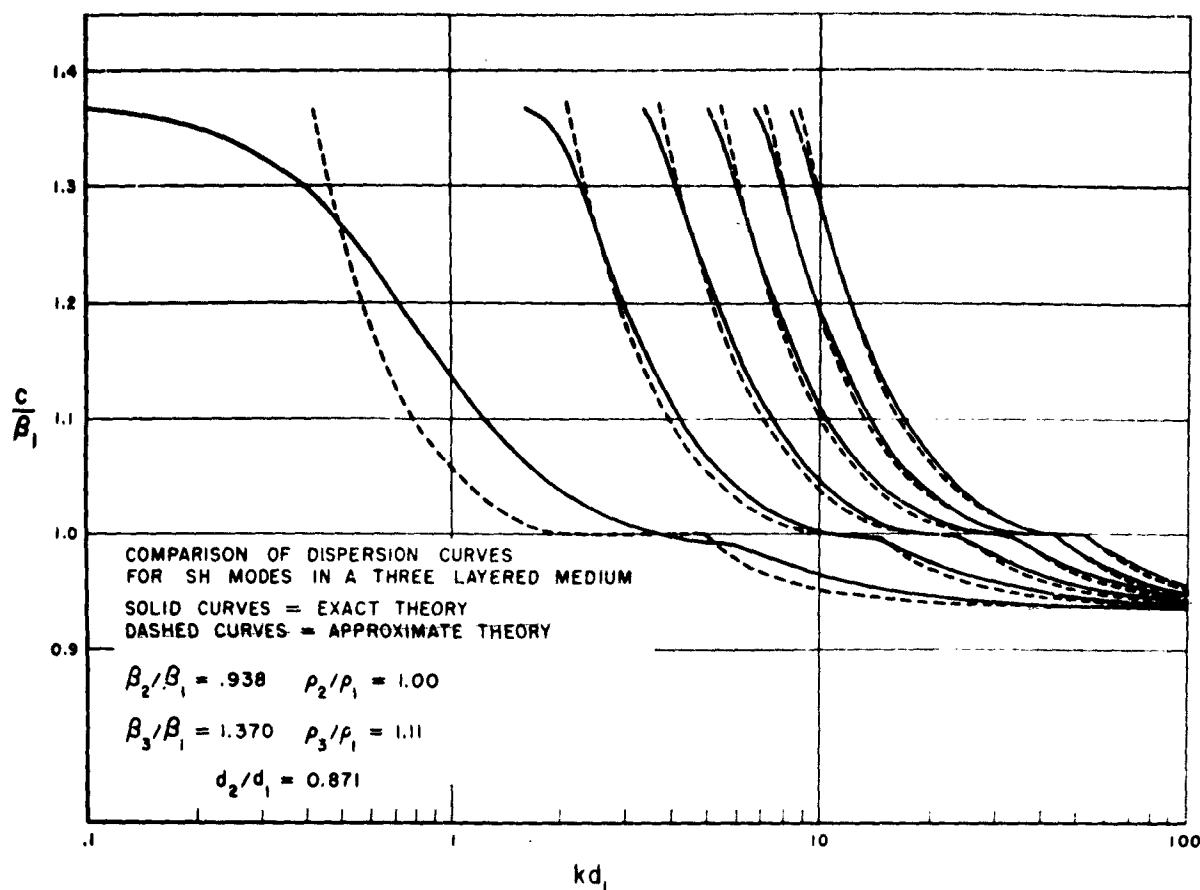
Fig. 6. Comparison of dispersion curves for *SH* modes in a three layered medium, Case II.

From these comparisons, it appears that the approximate theory gives a fairly good representation of the phase velocity dispersion curves for the normal modes of order higher than the 5th or 6th, except in the immediate neighborhood of the low frequency cut offs. For the lower modes the approximation is not good enough for quantitative purposes. In particular, in studies of seismic surface waves one will usually want to compare an observed variation of group velocity, $U = d(ck)/dk$, with period, with the theoretical group velocity curves for the modes of order $m = 0$ or 1 computed for some assumed velocity stratification. It is quite evident that the approximate curves for the cases illustrated here would not be adequate for this purpose.

It is believed, however, that the approximate theory will be found to have some value for the purpose of qualitative estimation even for the modes of lowest order. For example, in the case shown in Fig. 3, we know that the correct curve for $m = 0$ should approach the Rayleigh velocity of the semi-infinite layer as k approaches zero, and from the asymptotic form of the exact theory for small values of k, we have an expression from which the behavior of this curve for $kd << 1$ can be computed without difficulty. We also

know that for $kd > 10$ the phase velocity of this mode will be very close to the Rayleigh velocity of the upper layer. With these limiting values fixed and knowing that the approximate curve approaches the correct curve for some intermediate value of c, the general form of the curve can be estimated within a moderately broad band of uncertainty. In the case under consideration, we could have inferred by such a process that the minimum group velocity would probably occur at some value of kd between 2 and 4. The actual minimum, obtained by a graphical method from the plotted "exact" curve, is at $kd = 3.05$. In more complex cases, where the labor of computing the dispersion curves by the exact theory may become prohibitive, even so loose an estimate as this may be useful in limiting the range of possible structural models that might be used to explain a given set of observed dispersion data.

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